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ON THE HOMOGENEOUS CONE $z^2 = (10k^2 - 10k + 14)x^2 + y^2$

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Abstract:

The homogeneous ternary quadratic equation given by $z^2 = (10k^2 - 10k + 14)x^2 + y^2$ is analysed for its non-zero distinct integer solutions through different methods. A few interesting properties between the solutions are presented. Also, formulae for generating sequence of integer solutions based on the given solution are presented.

Keywords: Ternary quadratic, Integer solutions, Homogeneous cone.

Introduction:

It is well known that the quadratic Diophantine equations with three unknowns (homogeneous or non-homogeneous) are rich in variety [1, 2]. In particular, the ternary quadratic Diophantine equations of the form $z^2 = Dx^2 + y^2$ are analysed for values of D=29,41,43,47, 53, 55, 61, 63, 67in [3-11]. In [12],the homogeneous cone represented by the ternary quadratic equation $z^2 = 74x^2 + y^2$ has been studied. This result motivated us for determining integer solutions to the homogeneous cone $z^2 = Dx^2 + y^2$ when D takes even values. In this communication, yet another interesting homogeneous ternary quadratic Diophantine equation given by $z^2 = (10k^2 - 10k + 14)x^2 + y^2$ is analysed for its non-zero distinct integer solutions through different methods. A few interesting properties between the solutions are presented. Also, formulas for generating sequence of integer solutions based on the given solution are presented.

METHODS OF ANALYSIS

The ternary quadratic equation to be solved for its integer solutions is

$$z^2 = (10k^2 - 10k + 14)x^2 + y^2 \tag{1}$$

We present below different methods of solving (1):

Method: 1

(1) Is written in the form of ratio as

$$\frac{z+y}{(10k^2-10k+14)x}=\frac{x}{z-y}=\frac{r}{s}, s\neq 0 \qquad (2)$$

which is equivalent to the system of double equations

$$(10k^2-10k+14)rx-sy-sz=0$$
$$sx+ry-rz=0$$

Applying the method of cross-multiplication to the above system of equations,

$$x=x(r,s)=2rs$$
$$y=y(r,s)=(10k^2-10k+14)r^2-s^2$$
$$z=z(r,s)=(10k^2-10k+14)r^2+s^2$$

which satisfy (1)

Note: 1

It is observed that (1) may also be represented in the form of ratio as below:

(i) $\frac{z+y}{2x}=\frac{(5k^2-5k+7)x}{z-y}=\frac{r}{s}, s\neq 0$

The corresponding solutions to (1) are given as:

$$x=2rs, y=2r^2-(5k^2-5k+7)s^2, z=2r^2+(5k^2-5k+7)s^2$$

(ii) $\frac{z+y}{(5k^2-5k+7)x}=\frac{2x}{z-y}=\frac{r}{s}, s\neq 0$

The corresponding solutions to (1) are given as:

$$x=2rs, y=(5k^2-5k+7)r^2-2s^2, z=(5k^2-5k+7)r^2+2s^2$$

Method: 2

(1) Is written as the system of double equation in Table 1 as follows:

International Research Journal of Education and Technology
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Table: 1 System of Double Equations

System	1	2	3	4
z+y	$(10k^2-10k+14)x^2$	$(5k^2-5k+7)x^2$	$(10k^2-10k+14)x$	$(5k^2-5k+7)x$
z-y	1	2	x	$2x$

Solving each of the above system of double equations, the value of $x, y \,\&\, z$ satisfying (1) are obtained. For simplicity and brevity, in what follows, the integer solutions thus obtained are exhibited.

Solutions for system: I

No integer Solutions

Solutions for system: II

$$x=2s,\ y=2s^2(5k^2-5k+7)-1,\ z=2s^2(5k^2-5k+7)+1$$

Solution for system: III

$$x=2s,\ y=(10k^2-10k+13)s,\ z=(10k^2-10k+15)s$$

Solution for system: IV

$$x=2s,\ y=s(5k^2-5k+7)-2s,\ z=s(5k^2-5k+7)+2s$$

Method: 3

(1) Is written as

$$y^2+(10k^2-10k+14)x^2=z^2=z^2*1 \quad (3)$$

Assume z as

$$z=a^2+(10k^2-10k+14)b^2 \quad (4)$$

Write 1 as

International Research Journal of Education and Technology
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$$1=\frac{\left[\left(10k^2-10k+14\right)r^2-s^2+i2rs\sqrt{10k^2-10k+14}\right]\left[\left(10k^2-10k++14\right)r^2-s^2-i2rs\sqrt{10k^2-10k+14}\right]}{\left(\left(10k^2-10k+14\right)r^2+s^2\right)^2} \quad (5)$$

Using (4) & (5) in (3) and employing the method of factorization, consider

$$y+i\sqrt{10k^2-10k+14}\,x=\frac{\left(a+ib\sqrt{10k^2-10k+14}\right)^2\left[\left(10k^2-10k+14\right)r^2-s^2+i\sqrt{10k^2-10k+14}\,2rs\right]}{\left(10k^2-10k+14\right)r^2+s^2}$$

Equating real & imaginary parts, it is seen that

$$\left.\begin{aligned} y&=\frac{\left(a^2-\left(10k^2-10k+14\right)b^2\right)\left(\left(10k^2-10k+14\right)r^2-s^2\right)-4abrs\left(10k^2-10k+14\right)}{\left(10k^2-10k+14\right)r^2+s^2}\\ x&=\frac{\left(a^2-\left(10k^2-10k+14\right)b^2\right)2rs+2ab\left(\left(10k^2-10k+14\right)r^2-s^2\right)}{\left(10k^2-10k+14\right)r^2+s^2}\end{aligned}\right\} \quad (6)$$

Since our interest is to find the integer solutions, replacing a by $\left[\left(10k^2-10k+14\right)r^2+s^2\right]A$ & b by $\left[\left(10k^2-10k+14\right)r^2+s^2\right]B$ in (6) & (4), the corresponding integer solutions to (1) are given by

$$x=x(A,B)=\left(\left(10k^2-10k+14\right)r^2+s^2\right)\left[\left(A^2-\left(10k^2-10k+14\right)B^2\right)2rs+2AB\left(\left(10k^2-10k+14\right)r^2-s^2\right)\right]$$

$$y=y(A,B)=\left(\left(10k^2-10k+14\right)r^2+s^2\right)\left[\begin{aligned}\left(A^2-\left(10k^2-10k+14\right)B^2\right)\left[\left(10k^2-10k+14\right)r^2-s^2\right]\\ -4ABrs\left(10k^2-10k+14\right)\end{aligned}\right]$$

$$z=z(A,B)=\left(\left(10k^2-10k+14\right)r^2+s^2\right)^2\left(A^2+\left(10k^2-10k+14\right)B^2\right)$$

Following the above procedure,one may obtain difference sets of integer solutions to (1).

Method: 4

(1) Is written as

$$z^2-\left(10k^2-10k+14\right)x^2=y^2=y^2*1 \quad (7)$$

Assume y as

$$y = a^2 - (10k^2 - 10k + 14)b^2 \tag{8}$$

Write 1 as

$$1 = \frac{\left((10k^2 - 10k + 14)r^2 + s^2 + \sqrt{10k^2 - 10k + 14}\,2rs\right)\left((10k^2 - 10k + 14)r^2 + s^2 - \sqrt{10k^2 - 10k + 14}\,2rs\right)}{\left((10k^2 - 10k + 14)r^2 - s^2\right)^2} \tag{9}$$

Using (8) & (9) in (7) and employing the method of factorization, consider

$$z + \sqrt{10k^2 - 10k + 14}\,x = \frac{\left(a + \sqrt{10k^2 - 10k + 14}\,b\right)^2 \left((10k^2 - 10k + 14)r^2 + s^2 + 2rs\sqrt{10k^2 - 10k + 14}\right)}{(10k^2 - 10k + 14)r^2 - s^2}$$

Equating rational and irrational parts, it is seen that,

$$\begin{aligned} x &= \frac{\left(a^2 + (10k^2 - 10k + 14)b^2\right)2rs + 2ab\left((10k^2 - 10k + 14)r^2 + s^2\right)}{(10k^2 - 10k + 14)r^2 - s^2} \\ z &= \frac{\left(a^2 + (10k^2 - 10k + 14)b^2\right)\left((10k^2 - 10k + 14)r^2 + s^2\right) + 4abrs(10k^2 - 10k + 14)}{(10k^2 - 10k + 14)r^2 - s^2} \end{aligned} \tag{10}$$

Since our interest to find the integer solution, replacing a by $\left((10k^2 - 10k + 14)r^2 - s^2\right)\mathrm{A}$ & b by $\left((10k^2 - 10k + 14)r^2 - s^2\right)\mathrm{B}$ in (10) & (8), the corresponding integer solutions to (1) are given by

$$\begin{aligned} x = x(\mathrm{A},\mathrm{B}) &= \left((10k^2 - 10k + 14)r^2 - s^2\right)\left[\left(\mathrm{A}^2 + (10k^2 - 10k + 14)\mathrm{B}^2\right)2rs + 2\mathrm{AB}\left((10k^2 - 10k + 14)r^2 + s^2\right)\right] \\ y = y(\mathrm{A},\mathrm{B}) &= \left((10k^2 - 10k + 14)r^2 - s^2\right)^2\left[\mathrm{A}^2 - (10k^2 - 10k + 14)\mathrm{B}^2\right] \\ z = z(\mathrm{A},\mathrm{B}) &= \left((10k^2 - 10k + 14)r^2 - s^2\right)\left[\begin{array}{r}\left(\mathrm{A}^2 + (10k^2 - 10k + 14)\mathrm{B}^2\right)\left((10k^2 - 10k + 14)r^2 + s^2\right) \\ + 4ABrs\,(10k^2 - 10k + 14)\end{array}\right] \end{aligned}$$

Following the above procedure,one may obtain difference sets of integer solutions to (1).

GENERATION OF SOLUTIONS

Different formulas for generating sequence of integer solutions based on the given solution are presented below:

Let $(x_0, y_{0,} z_0)$ be any given solution to (1)

Formula: 1

Let $(x_{1,} y_1, z_1)$ given by

$$x_1 = 3x_0, y_1 = 3y_0 + h, z_1 = 3z_0 + 2h \tag{11}$$

be the 2^{nd} solution to (1). Using (11) in (1) and simplifying, one obtains

$$h = 2y_0 - 4z_0$$

In view of (11), the values of y_1 and z_1 are written in the matrix form as

$$(y_1, z_1)^t = M(y_0, z_0)^t$$

where

$$M = \begin{bmatrix} 5 & -4 \\ 4 & -5 \end{bmatrix}$$

and t is the transpose

The repetition of the above proses leads to the n^{th} solutions y_n, z_n givenby

$$(y_n, z_n)^t = M^n(y_0, z_0)^t$$

If α, β are the distinct eigen values of M, then

$$\alpha = 3, \beta = -3$$

We know that

$$M^n = \frac{a^n}{(\alpha-\beta)}(M-\beta I) + \frac{\beta^n}{(\beta-\alpha)}(M-\alpha I),\ I = 2\times 2 \text{ Identity matrix}$$

Thus, the general formulas for integer solutions to (1) are given by

$$x_n = 3^n x_0$$

$$\begin{pmatrix} y_n \\ z_n \end{pmatrix} = \frac{1}{3}\begin{bmatrix} 4\alpha^n - \beta^n & -2\alpha^n + 2\beta^n \\ 2\alpha^n - 2\beta^n & -\alpha^n + 4\beta^n \end{bmatrix}\begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$

Formula: 2

Let (x_1, y_1, z_1)given by

$$x_1 = h - (10k^2 - 10k + 15)x_0,\ y_1 = h - (10k^2 - 10k + 15)y_0,\ z_1 = (10k^2 - 10k + 15)z_0 \tag{12}$$

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be the 2^{nd} solution to (1). Using (12) in (1) and simplifying, one obtains

$$h=\left(20k^2-20k+28\right)x_0+2y_0$$

In view of (12), the values of x_1 and y_1 are written in the matrix form as

$$\left(x_1,y_1\right)^t=M\left(x_0,y_0\right)^t$$

Where $M=\begin{bmatrix}10k^2-10k+13 & 2\\ 20k^2-20k+28 & -\left(10k^2-10k+13\right)\end{bmatrix}$

And t is the transpose

The repetition of the above process leads to the n^{th} solutions x_n, y_n givenby

$$\left(x_n,y_n\right)^t=M^n\left(x_o,y_0\right)^t$$

If α,β arethe distinct eigen values of M, then

$$\alpha=10k^2-10k+15\,,\beta=-\left(10k^2-10k+15\right)$$

Thus, the general formulas for integer solutions to (1) are given by

$$\begin{pmatrix}x_n\\ y_n\end{pmatrix}=\frac{1}{\left(10k^2-10k+15\right)}\begin{bmatrix}\left(10k^2-10k+14\right)\alpha^n+\beta^n & \alpha^n-\beta^n\\ \left(10k^2-10k+14\right)\left(\alpha^n-\beta^n\right) & \alpha^n+\left(10k^2-10k+14\right)\beta^n\end{bmatrix}\begin{bmatrix}x_0\\ y_0\end{bmatrix}$$

$$z_n=\left(10k^2-10k+15\right)^n z_0$$

Formula: 3

Let $\left(x_1,y_1z_1\right)$ given by

$$x_1=h-\left(k^2+2k+10\right)x_0\ ,\ \ y_1=\left(k^2+2k+10\right)y_0,\ z_1=\left(k^2+2k+10\right)z_0+\left(3k-2\right)h \qquad (13)$$

be the 2^{nd} solution to (1). Using (13) in (1) and simplifying, one obtains

$$h=\left(20k^2-20k+28\right)x_0+\left(6k-4\right)z_0$$

In view of (13), the valuesof x_1 and z_1 are written in the matrix form as

$$\left(x_1,z_1\right)^t=M\left(x_0,z_0\right)^t$$

Where M= $\begin{bmatrix} 19k^2-22k+18 & 6k-4 \\ 60k^3-100k^2+124k-56 & 19k^2-22k+18 \end{bmatrix}$

and t is the transpose

The repetition of the above process leads to the n^{th} solutions x_n, z_n given by

$$(x_n, z_n)^t = M^n (x_0, z_0)^t$$

If α, β are the distinct eigen values of M, then

$$\alpha = 19k^2 - 22k + 18 + (6k-4)\sqrt{10k^2 - 10k + 14},$$
$$\beta = 19k^2 - 22k + 18 - (6k-4)\sqrt{10k^2 - 10k + 14}$$

Thus, the general formulas for integer solutions to (1) are given by

$$y_n = (k^2 + 2k + 10)^n y_0$$

$$\begin{pmatrix} x_n \\ z_n \end{pmatrix} = \begin{bmatrix} \dfrac{\alpha^n + \beta^n}{2} & \dfrac{\alpha^n - \beta^n}{2\left(\sqrt{10k^2 - 10k + 14}\right)} \\ \dfrac{(\alpha^n - \beta^n)\sqrt{(10k^2 - 10k + 14)}}{2} & \dfrac{\alpha^n + \beta^n}{2} \end{bmatrix} \begin{bmatrix} x_0 \\ z_0 \end{bmatrix}$$

Conclusion:

In this paper, an attempt has been made to obtain non-zero distinct integer solutions to the ternary quadratic Diophantine equation $z^2 = (10k^2 - 10k + 14)x^2 + y^2$ representing homogeneous cone. As there are varieties of cones, the readers may search for other forms of cones to obtain integer solutions for the corresponding cones.

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